SHENTON COLLEGE

Mathematics Specialist: Units 3 & 4

Test 1: Complex Numbers

Working time: 50 minutes
Total marks: 60 marks

Formula sheet provided
No notes permitted
No ClassPad (nor any other calculator) permitted

/ 60

Name: Teacher: ALFONSI MOORE

Note: Please read all questions carefully, and note that when a part of a question is worth more than two marks, adequate and clear working out is required for full marks.

1. Given that $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)$, $z_2 = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$, $z_3 = -2i$ and $z_4 = -1 - \sqrt{3}i$, determine

[1 + 2 + 2 + 3 + 3 = 11 marks]

(a) z_4 in polar form.

(b) $z_2 + z_3$ in Cartesian form.

(c) the product $z_1 z_2$ in polar form.

(d) the quotient $\frac{z_1}{z_3}$ in polar form.

(e) $(z_4)^6$ in Cartesian form.

2. If $z = r\text{cis}\theta$, express the following in cis form in terms of r and/or θ:

[1 + 1 + 1 + 1 = 4 marks]

(a) $\bar{z}$

(b) $z\bar{z}$

(c) iz^2

(d) $\dfrac{1-i}{1+i}z$

3. Arithmetic operations on complex numbers can be described geometrically in terms of *translations*, *rotations*, *reflections* and *enlargements* in the complex plane.

Explain the sequence of transformations which correspond to taking a complex number z and transforming it to $2i(\bar{z} - i)$.

[4 marks]

4. Consider the equation $z^4 = 2\sqrt{3} + 2i$.

[5 + 1 = 6 marks]

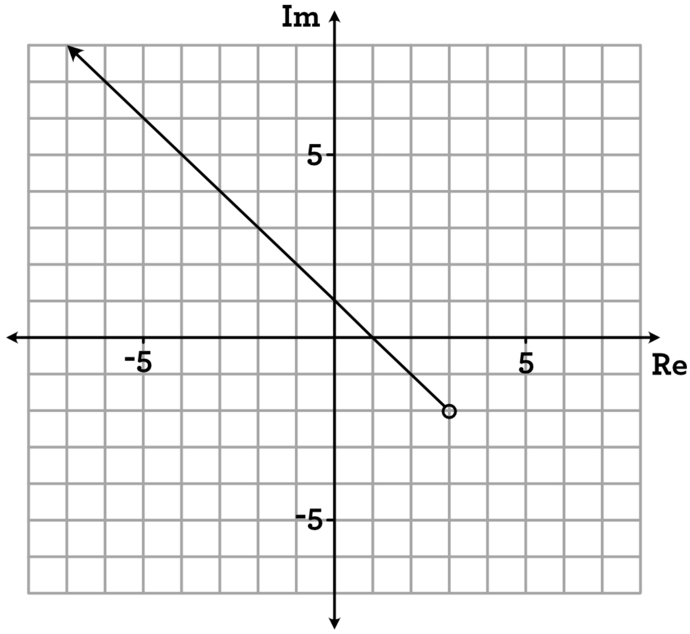
(a) Determine all of the solutions to this equation, giving your answers in polar form and in terms of their prinicpal argument.

(b) Explain in a single sentence why it is unsurprising that none of the solutions of the polynomial $z^4 - (2\sqrt{3} + 2i) = 0$ are complex conjugate pairs.

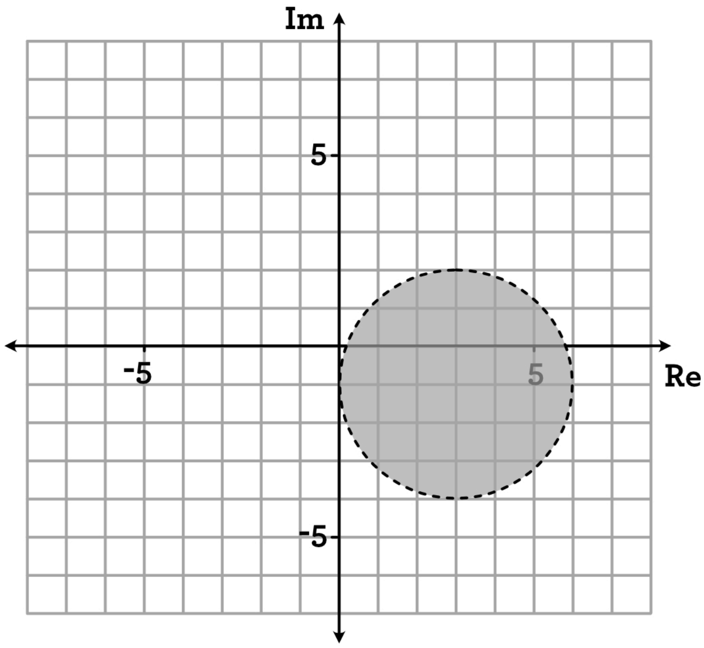
5. Express, using set notation, the locus of z in each of the following diagrams.

[3 + 3 = 6 marks]

(a)



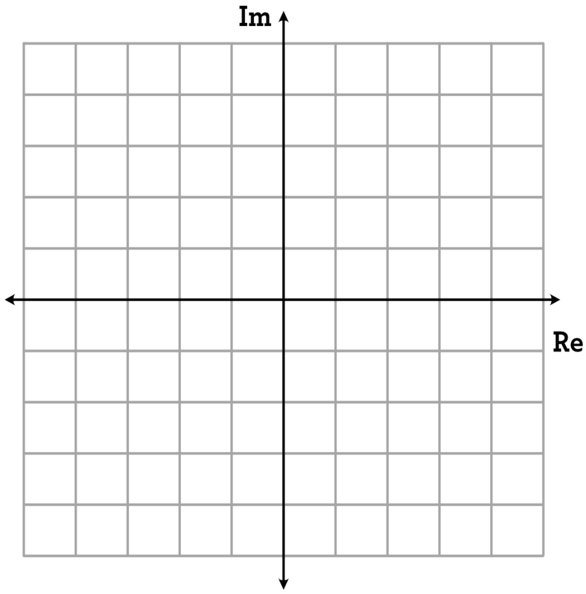
(b)



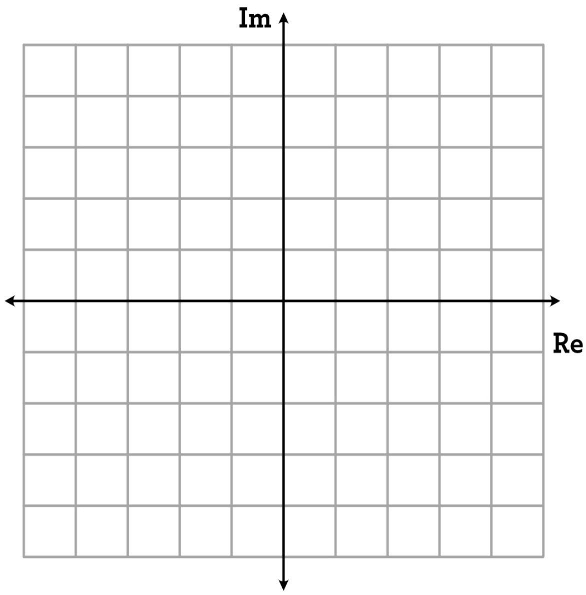
6. Use the Argand diagrams provided to sketch the regions in the complex plane defined by the following loci.

[3 + 3 + 3 = 9 marks]

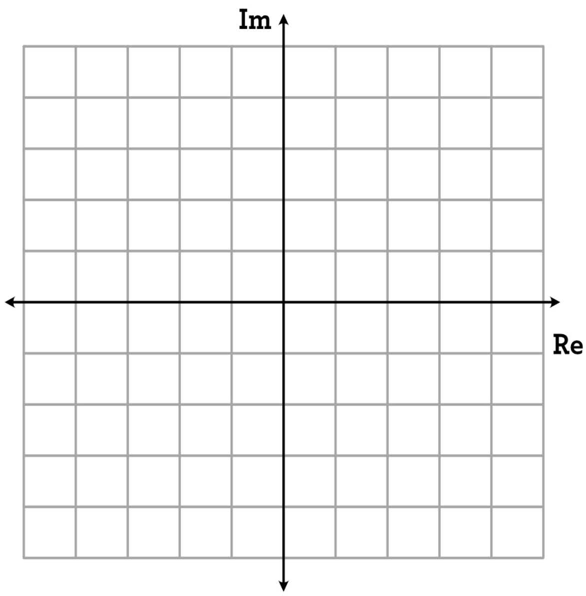
(a) $\{z : |z - 1 + i| = |z + 1 - 3i|\}$



(b) $\{z : \frac{\pi}{6} \leq \arg(z) \leq \frac{\pi}{4}\} \cap \{z : 1 \leq \text{Re}(z) \leq 4\}$



(c) $\{z : \text{Im}(z) = |z - i|\}$



7. Consider $P(z) = z^3 + az^2 + bz + c$ with $a, b, c \in \mathbb{R}$. Two of the roots of $P(z) = 0$ are -2 and $(-3 + 2i)$.

[2 + 3 = 5 marks]

(a) Write $P(z)$ in fully-factored form. (I.e., express $P(z)$ as the product of its linear factors.)

(b) Hence, determine the values of the coefficients a, b and c.

8. Given that

$$\frac{\sin 4\theta}{\sin \theta} = A\cos^3\theta + B\cos\theta \qquad (\sin\theta \neq 0)$$

[5 + 1 = 6 marks]

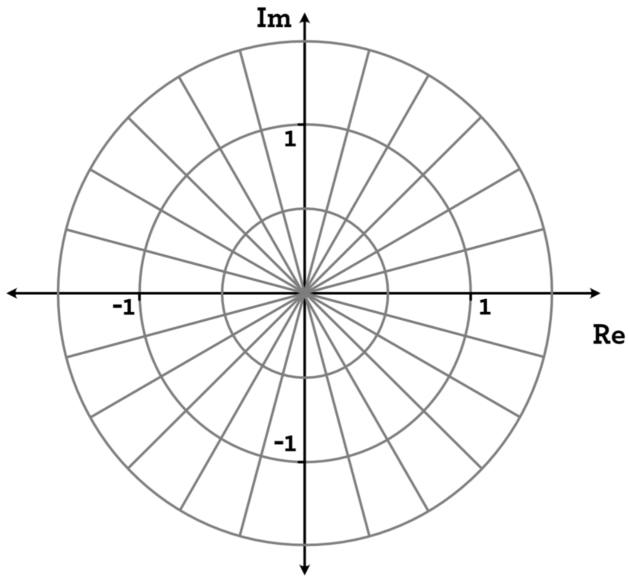
(a) use de Moivre's theorem with $n = 4$ to determine the values of A and B.

(b) Hence, determine the limiting value of $\frac{\sin 4\theta}{\sin \theta}$ as θ approaches zero (i.e., $\lim_{\theta\to 0} \frac{\sin 4\theta}{\sin \theta}$).

9. Consider $u = \text{cis}\left(\frac{\pi}{4}\right)$, one of the 8th roots of unity, and $v = \text{cis}\left(\frac{\pi}{3}\right)$, one of the 6th roots of unity.

[2 + 2 + 2 + 3 = 9 marks]

(a) Mark and label the positions of u, u^2, u^4 and u^6 as well as v, v^2, v^4 and v^6 on the Argand diagram at the right.



(b) For what values of $m \in \mathbb{Z}$ is u^m purely real and negative?

(c) For what values of $n \in \mathbb{Z}$ is v^n purely real and positive?

(d) What is the smallest value of $p \in \mathbb{Z}^+$ such that the product $u^{2-p}v^{p-7}$ is purely real?

[END OF TEST]